On the Statistical Distribution of Spacecraft Maximum Structural Response

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In most aerospace engineering applications, the finite number of flight data accumulated in the past is not sufficient to characterize nonstationary random excitations resulting from each flight event, such as booster engine ignition or burnout. In this article, a direct statistical analysis of spacecraft maximum structural response is performed and the spacecraft structural reliability is obtained. It is found that the Gumbel Type I asymptotic distribution of maximum values provides a reasonably good statistical model for spacecraft maximum structural responses. The current approach makes it possible to perform the reliability-based optimum design of spacecraft structures.

Introduction

Flight data taken on a number of past flights indicate that major spacecraft excitations and responses resulting from booster engine ignition or booster engine shutdown are nonstationary random processes (e.g., References 1 and 2). These excitations not only are highly transient but may also produce some of the most severe spacecraft vibrations in the entire mission. A typical example taken from Reference 2 is given in Figure 1 in which three sample functions of nonstationary random processes applied at the base of the launch vehicle are given. In Figure 1, two random forcing processes resulting from two booster engines have been converted into a resultant thrust process $F(t)$ and a bending moment process $M(t)$.

For the reliability prediction and the design of spacecraft structures, the rigorous random vibration approach, unfortunately, suffers from a number of difficulties as follows:

(1) The number of sample functions accumulated from past flights for the same type of booster engine is not sufficient for the statistical characterizations of $F(t)$ and $M(t)$, as well as their correlations. Thus far, 27 samples are available.

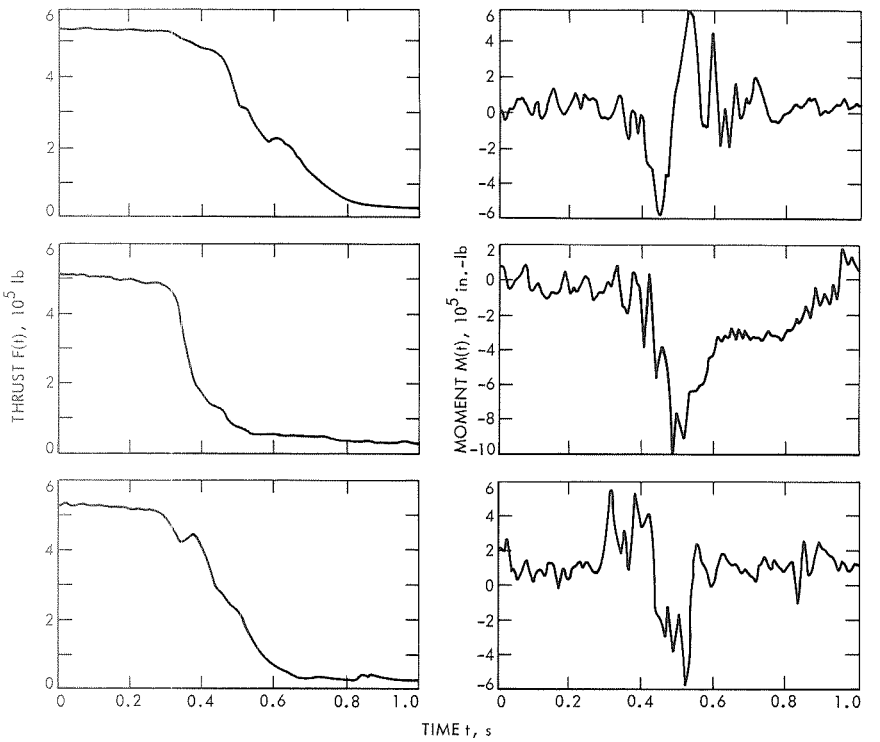

Figure 1. Typical sample function of nonstationary random excitation (from Reference 2)

(2) Because of the complexity of the entire space vehicle structure and the excessive computational cost involved, the redesign of the spacecraft structure is usually made separately from the vehicle, using the resulting six random acceleration processes, $g_i(t), i = 1,2,\cdots,6$, at the spacecraft/boost vehicle (S/BV) interface as input excitations for approximation (Reference 1). Therefore, even if the number of samples is sufficient to characterize random excitations $F(t)$ and $M(t)$, the computer storage space required for all the digitized data of covariance functions $K_{g_i g_j}(t_1, t_2)$, or generalized spectra $S_{g_i g_j}(\omega_1, \omega_2)$, of all the six random processes, $g_i(t)$, $i = 1,2,\cdots,6$, may easily exceed the computer capacity. Furthermore, the numerical operations involved in computing the statistics of responses and response derivatives, such as covariance functions, are quite involved. Preliminary investigation indicates that the cost of spacecraft redesign (many cycles of redesign) becomes intolerable when the rigorous random vibration approach is employed.

(3) The prediction of structural reliability or failure, such as the first excursion probability, under nonstationary random excitations is still in the primitive stage. Hence, the final reliability figure usually

involves two kinds of approximations: (a) the approximation of known random processes, such as gaussian processes, to the actual processes, and (b) the approximation made in estimating the reliability from the response statistics.

It is due to these theoretical, as well as practical, difficulties that other possible approaches appear to be highly desirable. In fact, some attempts have been made from the viewpoint of deterministic analysis. The premise of deterministic analysis approaches, so far, is to obtain an upper bound of the maximum response under a specified class of excitations. In this connection, the upper bound of the maximum structural response based on the excitation energy (Reference 3), the response envelope in the frequency domain analysis (References 1 and 4), and the response envelope in the time domain analysis (Reference 5) should be mentioned. Although the foregoing deterministic analyses provide useful information for structural design, there are two inherent discrepancies: (1) the upper bound obtained may be too high and there is no indication of how conservative it may be, and (2) the estimated upper bound is not associated with any probability statement. Since the excitations are random processes, the probability statement is important in design.

It has been realized that the source of randomness in excitations is due to the fact that the properties of rocket motors are associated with considerable statistical variation. Thus, a sample of a rocket motor generates a sample of random excitations which, in turn, produces a sample value of maximum response at certain points of the spacecraft. Hence, there is a one-to-one correspondence between the maximum spacecraft response and the property of the rocket motor. Since the underlying physical process describing how the ignition or shutdown of the random rocket motor is related to the random excitation processes is not clear, the assumption of certain types of nonstationary random processes, such as random processes obtained by passing a modulated short noise through a filter (Reference 1), is difficult to be justified. It, therefore, appears that a direct statistical analysis of the spacecraft maximum response may be a reasonable approach when 27 sample functions are available.

Statistical Analysis of Maximum Response

The advantages of the direct statistical analysis of spacecraft maximum responses can be summarized as follows: (1) the cost of analysis and redesign is within practical limitation, (2) the accuracy of the reliability estimate increases as the number of samples increase, and (3) it admits to a reliability-based optimum design of spacecraft structures.

Let the structural system of the space vehicle be characterized by a linear time invariant operator L, which relates the spacecraft stress

response vector $\mathbf{Y}(t)$, consisting of component $\mathbf{Y}_k(t)$, $k = 1,2,\cdots$, to the excitation vector process $\mathbf{X}(t)$ through the following equation:

$$L[\mathbf{Y}(t)] = \mathbf{X}(t) \tag{1}$$

where $\mathbf{X}(t)$ consists of the thrust process $F(t)$ and the bending moment process $M(t)$ at the base of the launch vehicle, i.e.,

$$\mathbf{X}(t) = [F(t)\ M(t)]'$$

When the redesign of spacecraft structures is separated from the space vehicle, L represents the spacecraft characteristics and

$$\mathbf{X}(t) = [g_1(t)\ g_2(t) \cdots g_6(t)]'$$

is the acceleration vector process at the S/BV interface.

The stress response vector $\mathbf{Y}(t)$ can be written as

$$\mathbf{Y}(t) = \int_0^t \mathbf{h}(t - \tau)\,\mathbf{X}(\tau)\,d\tau \tag{2}$$

or

$$\mathbf{Y}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}(\omega)\mathbf{X}(\omega)e^{-i\omega t}d\omega \tag{3}$$

in which $\mathbf{h}(t)$ and $\mathbf{H}(\omega)$ are, respectively, the impulse response matrix and the frequency response matrix or transfer matrix. Both $\mathbf{h}(t)$ and $\mathbf{H}(\omega)$ are Fourier transform pairs, and they are functions of natural frequencies, modes, generalized masses, and dampings of the structure (e.g., Reference 1). $\mathbf{X}(\omega)$ is the Fourier transform of $\mathbf{X}(t)$. The solution of Equation 2 is referred to as the time domain analysis, while the solution of Equation 3 is referred to as the frequency domain analysis.

Let $\mathbf{X}^j(t)$, $\mathbf{X}^j(\omega)$, and $\mathbf{Y}_k^j(t)$ represent the jth samples of $\mathbf{X}(t)$, $\mathbf{X}(\omega)$, and $\mathbf{Y}_k(t)$, respectively, where $\mathbf{Y}_k(t)$ is the spacecraft stress response at point k. The absolute maximum, $\mathbf{Z}$, of $\mathbf{Y}_k(t)$ for $t \geq 0$ is a random variable whose jth sample value $\mathbf{Z}_j$ follows from Equations 2 and 3 as

$$\mathbf{Z}_j = \max_t \left| \mathbf{Y}_k^j(t) \right| = \max_t \left| \int_0^t \mathbf{h}_k(t - \tau)\mathbf{X}^j(\tau)\,d\tau \right| \tag{4a}$$

$$= \frac{1}{2\pi} \max_t \left| \int_{-\infty}^{\infty} \mathbf{H}_k(\omega)\mathbf{X}^j(\omega)e^{-i\omega t}\,d\omega \right| \tag{4b}$$

where $\mathbf{H}_k(\omega)$ and $\mathbf{h}_k(t)$ are the kth row of $\mathbf{H}(\omega)$ and $\mathbf{h}(t)$, respectively.

Then, rearrange $\mathbf{Z}_j$, $j = 1,2, \cdots, 27$, in an ascending order such that $\mathbf{Z}_1 \leq \mathbf{Z}_2 \cdots \leq \mathbf{Z}_{27}$.

Some investigations (Reference 6) indicate that when the excitation $\mathbf{X}(t)$ is nonstationary gaussian, the distribution of the peaks or troughs within a time interval $[0,T]$ is approximately the Weibull distribution, which is a distribution of exponential type. Furthermore, large peaks or troughs, being of particular interest in design, are approximately independent; and if the number of peaks or troughs in the response history approaches infinity, the asymtotic distribution of the maximum response $\mathbf{Z}$ can be shown (References 6 and 7) to follow the Gumbel Type I asymptotic distribution of maximum values as follows:

$$F_{\mathbf{Z}}(x) = \Pr[\mathbf{Z} \leq x] = \exp\{-\exp[-\sigma(x - \mu)]\}$$
$$\sigma > 0, -\infty \leq x \leq \infty, -\infty \leq \mu \leq \infty \tag{5}$$

where $F_{\mathbf{Z}}(x)$ is the distribution function of the maximum stress response $\mathbf{Z}$, and σ and μ are distribution parameters.

If the samples $\mathbf{Z}_1,\mathbf{Z}_2, \cdots, \mathbf{Z}_{27}$ follow the distribution of Equation 5, they should be scattered around a straight line when plotted on Gumbel extreme-value probability paper. The empirical distribution function $F_m(Z_k)$ for the sample values $\mathbf{Z}_1,\mathbf{Z}_2, \cdots, \mathbf{Z}_{27}$ can be established as

$$F_m(Z_k) = (k - 0.5)/n \tag{6}$$

where n is the total number of samples and is equal to 27 herein.

The first quadrant of a preliminary design of a spacecraft is shown schematically in Figure 2 (four quadrants are symmetrical). Sample values of maximum stresses $\mathbf{Z}_1,\mathbf{Z}_2, \cdots, \mathbf{Z}_{27}$ in each structural member, due to 27 samples of excitation vector $\mathbf{X}(t)$ at the base of the launch vehicle during the stage 1 shutdown, are computed. These sample values are then plotted on both the Gumbel extreme-value probability paper and the cumulative probability paper; five of these typical plots are given in Figure 3. In Figure 3, members 1 and 2 are capsule members, members 3 and 4 are spacecraft members, and member 5 is a member supporting the fuel tank (see Figure 2). All the plots indicate that the Gumbel Type I asymptotic distribution of maximum values is a reasonably good model. Results of the Kolmogorov–Smirnov test (e.g., Reference 8) indicate that the hypothesis that the sample data of maximum stress responses in each member follow the distribution of Equation 5 can be accepted at least at the 20% level of significance.

Other distribution functions, such as normal, lognormal, Weibull, and gamma, etc., have been tried; it was found that the plots associated with

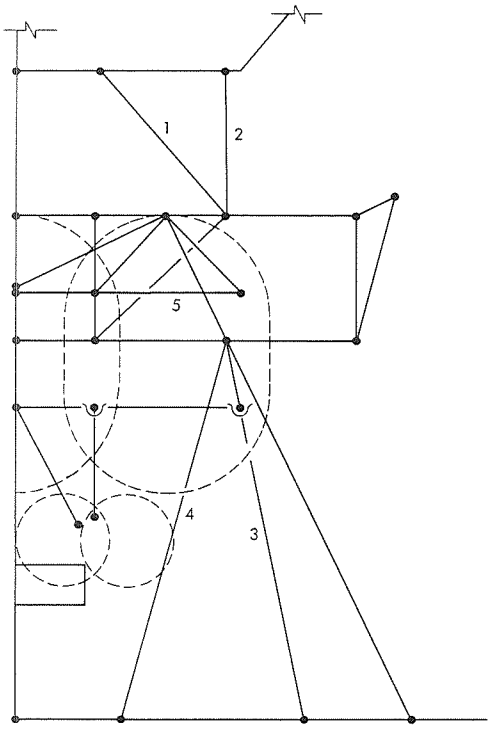

Figure 2. Schematic configuration of a spacecraft

these distributions exhibit a clear departure from linearity, and the associated Kolmogorov–Smirnov statistics are higher than those associated with the distribution of Equation 5. This is an indication that Equation 5 may be the best to fit the samples of maximum stress response. Indeed, the Gumbel Type I asymptotic distribution of maximum values (Equation 5) is a logical choice for the maximum response in approximation, although the conditions for the distribution that peaks are independent and that the number of peaks approaches infinity are not satisfied. In fact, some investigators (Reference 9) in earthquake engineering have shown that Equation 5 provides a reasonable statistical model for maximum structure response due to earthquakes, where the conditions for the distribution are clearly violated. Gumbel has also shown problems (Reference 7) where the distribution conditions are not satisfied but where the distribution of Equation 5 is a good model.

The probability of first excursion failure p_k of the kth structural member follows from Equation 5 as

$$p_k = \int_{-\infty}^{\infty} (1 - \exp\{-\exp[-\sigma_k(x - \mu_k)]\})\, f_k(x)\, dx \qquad (7)$$

where $f_k(x)$ is the probability density of the ultimate strength of the kth member, and σ_k and μ_k are the associated distribution parameters which

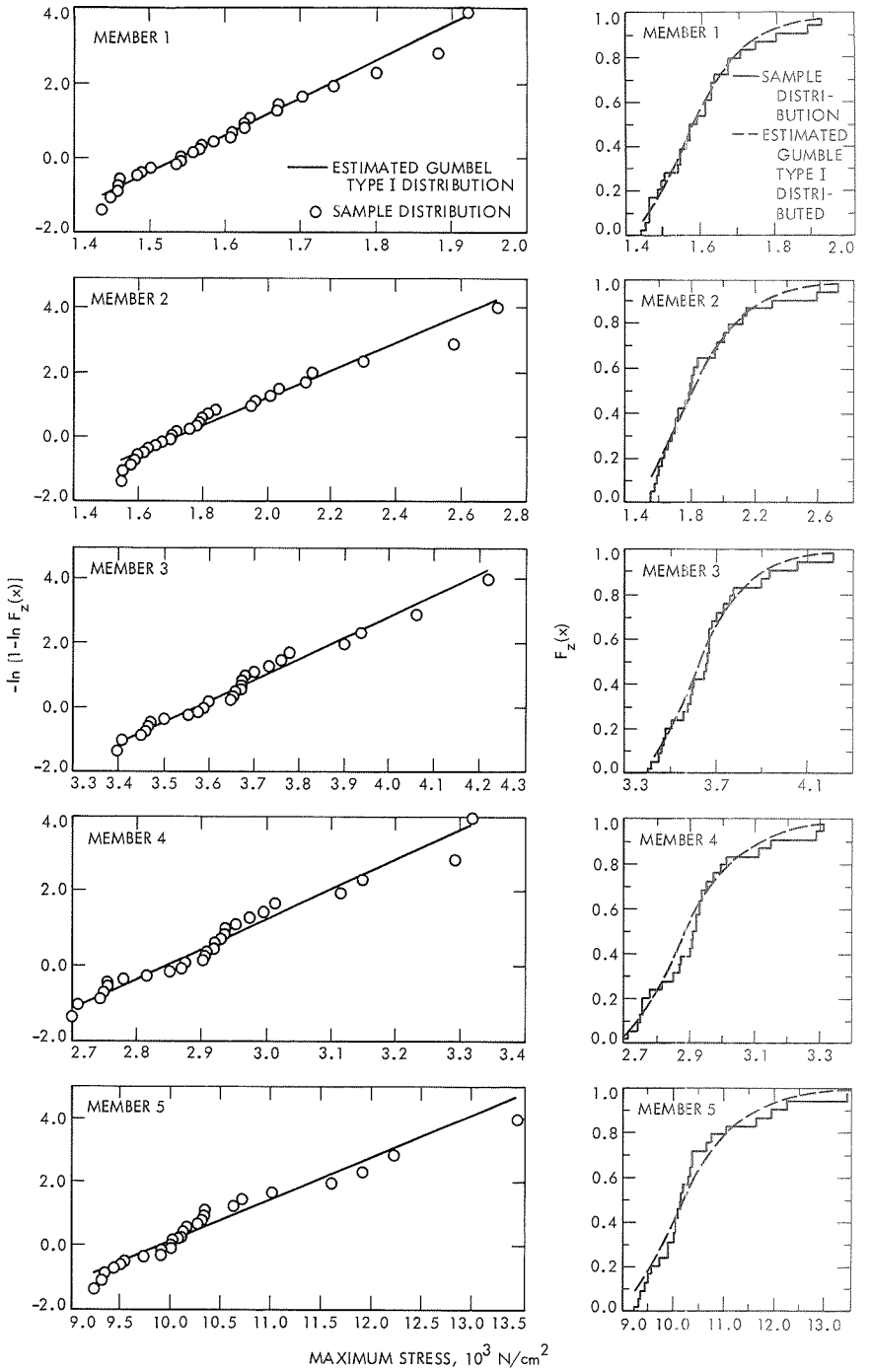

Figure 3. Distribution of maximum stress response

can easily be estimated from Figure 3. The material used for each structural member of the spacecraft (Figure 2) is aluminum with a mean ultimate strength of 20.68×10^3 N/cm^2. Results of this analysis indicate that the probability of failure of each member is negligibly small except member 5, which has a probability of failure 0.45×10^{-6}, when the statistical dispersion of the ultimate strength of aluminum is negligible.

It follows from Equation 4 that the computation of 27 samples of maximum stress response $\mathbf{Z}_j$, $j = 1,2, \cdots, 27$, for each structural member requires neither heavy computational effort nor excessive storage in computer. In fact, using the fast Fourier transform, $\mathbf{Z}_j$ can easily be computed (Equation 4b). Therefore, the cost involved in computing the structural reliability, Equation 7, as well as redesign of spacecraft structures, is within practical limitation.

While it is not difficult to compute structural reliability, one can proceed to perform the reliability-based optimum design of spacecraft structures (e.g., Reference 10), and minimize the structural weight subject to the constraint that the overall probability of failure of the spacecraft p_f should be less than certain value p_a. The feasibility of incorporating the notion of reliability in optimum design is another advantage of the current approach. It should be mentioned that much research work remains to be done before the rigorous nonstationary random vibration approach can be incorporated in optimum structural design.

Conclusion

Under nonstationary random excitations resulting from booster engine shutdown, a direct statistical analysis of spacecraft maximum responses is performed and the spacecraft structural reliability is obtained. It is found that the Gumbel Type I asymptotic distribution of maximum values provides a reasonably good statistical model for spacecraft maximum responses. The current approach makes it possible to perform the reliability-based optimum design of spacecraft structures.

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